11th edition

# ELEMENTARY LINEAR ALGEBRA

Howard Anton | Chris Rorres

APPLICATIONS VERSION

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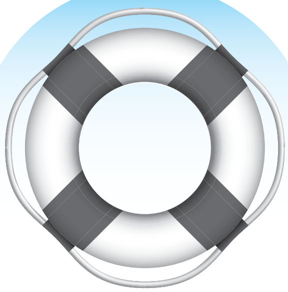

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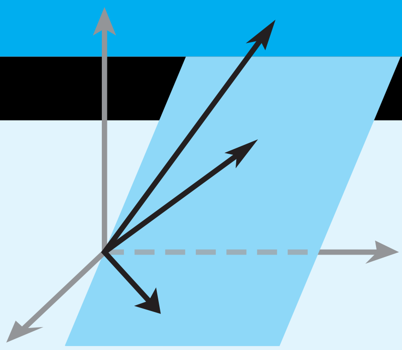

11TH EDITION

# Elementary Linear Algebra

## Applications Version


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WILEY

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# ABOUT THE AUTHOR

Howard Anton obtained his B.A. from Lehigh University, his M.A. from the University of Illinois, and his Ph.D. from the Polytechnic University of Brooklyn, all in mathematics. In the early 1960s he worked for Burroughs Corporation and Avco Corporation at Cape Canaveral, Florida, where he was involved with the manned space program. In 1968 he joined the Mathematics Department at Drexel University, where he taught full time until 1983. Since then he has devoted the majority of his time to textbook writing and activities for mathematical associations. Dr. Anton was president of the EPADEL Section of the Mathematical Association of America (MAA), served on the Board of Governors of that organization, and guided the creation of the Student Chapters of the MAA. In addition to various pedagogical articles, he has published numerous research papers in functional analysis, approximation theory, and topology. He is best known for his textbooks in mathematics, which are among the most widely used in the world. There are currently more than 175 versions of his books, including translations into Spanish, Arabic, Portuguese, Italian, Indonesian, French, Japanese, Chinese, Hebrew, and German. For relaxation, Dr. Anton enjoys travel and photography.

Chris Rorres earned his B.S. degree from Drexel University and his Ph.D. from the Courant Institute of New York University. He was a faculty member of the Department of Mathematics at Drexel University for more than 30 years where, in addition to teaching, he did applied research in solar engineering, acoustic scattering, population dynamics, computer system reliability, geometry of archaeological sites, optimal animal harvesting policies, and decision theory. He retired from Drexel in 2001 as a Professor Emeritus of Mathematics and is now a mathematical consultant. He also has a research position at the School of Veterinary Medicine at the University of Pennsylvania where he does mathematical modeling of animal epidemics. Dr. Rorres is a recognized expert on the life and work of Archimedes and has appeared in various television documentaries on that subject. His highly acclaimed website on Archimedes (http://www.math.nyu.edu/~crorres/Archimedes/contents.html) is a virtual book that has become an important teaching tool in mathematical history for students around the world.

To:
My wife, Pat

My children, Brian, David, and Lauren

My parents, Shirley and Benjamin

My benefactor, Stephen Girard (1750–1831),
whose philanthropy changed my life

*Howard Anton*

To:
Billie

*Chris Rorres*

# PREFACE

This textbook is an expanded version of *Elementary Linear Algebra*, eleventh edition, by Howard Anton. The first nine chapters of this book are identical to the first nine chapters of that text; the tenth chapter consists of twenty applications of linear algebra drawn from business, economics, engineering, physics, computer science, approximation theory, ecology, demography, and genetics. The applications are largely independent of each other, and each includes a list of mathematical prerequisites. Thus, each instructor has the flexibility to choose those applications that are suitable for his or her students and to incorporate each application anywhere in the course after the mathematical prerequisites have been satisfied. Chapters 1–9 include simpler treatments of some of the applications covered in more depth in Chapter 10.

This edition gives an introductory treatment of linear algebra that is suitable for a first undergraduate course. Its aim is to present the fundamentals of linear algebra in the clearest possible way—sound pedagogy is the main consideration. Although calculus is not a prerequisite, there is some optional material that is clearly marked for students with a calculus background. If desired, that material can be omitted without loss of continuity.

Technology is not required to use this text, but for instructors who would like to use MATLAB, *Mathematica*, Maple, or calculators with linear algebra capabilities, we have posted some supporting material that can be accessed at either of the following companion websites:

www.howardanton.com
www.wiley.com/college/anton

## Summary of Changes in This Edition

Many parts of the text have been revised based on an extensive set of reviews. Here are the primary changes:

- **Earlier Linear Transformations** Linear transformations are introduced earlier (starting in Section 1.8). Many exercise sets, as well as parts of Chapters 4 and 8, have been revised in keeping with the earlier introduction of linear transformations.
- **New Exercises** Hundreds of new exercises of all types have been added throughout the text.
- **Technology** Exercises requiring technology such as MATLAB, *Mathematica*, or Maple have been added and supporting data sets have been posted on the companion websites for this text. The use of technology is not essential, and these exercises can be omitted without affecting the flow of the text.
- **Exercise Sets Reorganized** Many multiple-part exercises have been subdivided to create a better balance between odd and even exercise types. To simplify the instructor's task of creating assignments, exercise sets have been arranged in clearly defined categories.
- **Reorganization** In addition to the earlier introduction of linear transformations, the old Section 4.12 on *Dynamical Systems and Markov Chains* has been moved to Chapter 5 in order to incorporate material on eigenvalues and eigenvectors.
- **Rewriting** Section 9.3 on *Internet Search Engines* from the previous edition has been rewritten to reflect more accurately how the Google PageRank algorithm works in practice. That section is now Section 10.20 of the applications version of this text.
- **Appendix A Rewritten** The appendix on reading and writing proofs has been expanded and revised to better support courses that focus on proving theorems.
- **Web Materials** Supplementary web materials now include various applications modules, three modules on linear programming, and an alternative presentation of determinants based on permutations.
- **Applications Chapter** Section 10.2 of the previous edition has been moved to the websites that accompany this text, so it is now part of a three-module set on Linear

Programming. A new section on Internet search engines has been added that explains the PageRank algorithm used by Google.

*Hallmark Features*

- **Relationships Among Concepts** One of our main pedagogical goals is to convey to the student that linear algebra is a cohesive subject and not simply a collection of isolated definitions and techniques. One way in which we do this is by using a crescendo of *Equivalent Statements* theorems that continually revisit relationships among systems of equations, matrices, determinants, vectors, linear transformations, and eigenvalues. To get a general sense of how we use this technique see Theorems 1.5.3, 1.6.4, 2.3.8, 4.8.8, and then Theorem 5.1.5, for example.
- **Smooth Transition to Abstraction** Because the transition from $R^n$ to general vector spaces is difficult for many students, considerable effort is devoted to explaining the purpose of abstraction and helping the student to "visualize" abstract ideas by drawing analogies to familiar geometric ideas.
- **Mathematical Precision** When reasonable, we try to be mathematically precise. In keeping with the level of student audience, proofs are presented in a patient style that is tailored for beginners.
- **Suitability for a Diverse Audience** This text is designed to serve the needs of students in engineering, computer science, biology, physics, business, and economics as well as those majoring in mathematics.
- **Historical Notes** To give the students a sense of mathematical history and to convey that real people created the mathematical theorems and equations they are studying, we have included numerous *Historical Notes* that put the topic being studied in historical perspective.

*About the Exercises*

- **Graded Exercise Sets** Each exercise set in the first nine chapters begins with routine drill problems and progresses to problems with more substance. These are followed by three categories of exercises, the first focusing on proofs, the second on true/false exercises, and the third on problems requiring technology. This compartmentalization is designed to simplify the instructor's task of selecting exercises for homework.
- **Proof Exercises** Linear algebra courses vary widely in their emphasis on proofs, so exercises involving proofs have been grouped and compartmentalized for easy identification. Appendix A has been rewritten to provide students more guidance on proving theorems.
- **True/False Exercises** The True/False exercises are designed to check conceptual understanding and logical reasoning. To avoid pure guesswork, the students are required to justify their responses in some way.
- **Technology Exercises** Exercises that require technology have also been grouped. To avoid burdening the student with keyboarding, the relevant data files have been posted on the websites that accompany this text.
- **Supplementary Exercises** Each of the first nine chapters ends with a set of supplementary exercises that draw on all topics in the chapter. These tend to be more challenging.

*Supplementary Materials for Students*

- **Student Solutions Manual** This supplement provides detailed solutions to most odd-numbered exercises (ISBN 978-1-118-464427).
- **Data Files** Data files for the technology exercises are posted on the companion websites that accompany this text.
- **MATLAB Manual and Linear Algebra Labs** This supplement contains a set of MATLAB laboratory projects written by Dan Seth of West Texas A&M University. It is designed to help students learn key linear algebra concepts by using MATLAB and is available in PDF form without charge to students at schools adopting the 11th edition of the text.
- **Videos** A complete set of Daniel Solow's *How to Read and Do Proofs* videos is available to students through WileyPLUS as well as the companion websites that accompany

this text. Those materials include a guide to help students locate the lecture videos appropriate for specific proofs in the text.

*Supplementary Materials for Instructors*

- **Instructor's Solutions Manual** This supplement provides worked-out solutions to most exercises in the text (ISBN 978-1-118-434482).
- **PowerPoint Presentations** PowerPoint slides are provided that display important definitions, examples, graphics, and theorems in the book. These can also be distributed to students as review materials or to simplify note taking.
- **Test Bank** Test questions and sample exams are available in PDF or LaTeX form.
- **WileyPLUS** An online environment for effective teaching and learning. WileyPLUS builds student confidence by taking the guesswork out of studying and by providing a clear roadmap of what to do, how to do it, and whether it was done right. Its purpose is to motivate and foster initiative so instructors can have a greater impact on classroom achievement and beyond.

*A Guide for the Instructor*

Although linear algebra courses vary widely in content and philosophy, most courses fall into two categories—those with about 40 lectures and those with about 30 lectures. Accordingly, we have created long and short templates as possible starting points for constructing a course outline. Of course, these are just guides, and you will certainly want to customize them to fit your local interests and requirements. Neither of these sample templates includes applications or the numerical methods in Chapter 9. Those can be added, if desired, and as time permits.

| | Long Template | Short Template |
|---|---|---|
| **Chapter 1:** Systems of Linear Equations and Matrices | 8 lectures | 6 lectures |
| **Chapter 2:** Determinants | 3 lectures | 2 lectures |
| **Chapter 3:** Euclidean Vector Spaces | 4 lectures | 3 lectures |
| **Chapter 4:** General Vector Spaces | 10 lectures | 9 lectures |
| **Chapter 5:** Eigenvalues and Eigenvectors | 3 lectures | 3 lectures |
| **Chapter 6:** Inner Product Spaces | 3 lectures | 1 lecture |
| **Chapter 7:** Diagonalization and Quadratic Forms | 4 lectures | 3 lectures |
| **Chapter 8:** General Linear Transformations | 4 lectures | 3 lectures |
| **Total:** | **39 lectures** | **30 lectures** |

*Reviewers*

The following people reviewed the plans for this edition, critiqued much of the content, and provided me with insightful pedagogical advice:

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**Roger Lipsett**, *Brandeis University* – who proofread the manuscript and exercise solutions for mathematical accuracy.

**Daniel Solow**, *Case Western Reserve University* – author of "*How to Read and Do Proofs*," for providing videos on techniques of proof and a key to using those videos in coordination with this text.

**Sky Pelletier Waterpeace** – who critiqued the technology exercises, suggested improvements, and provided the data sets.

**Special Contributions** I would also like to express my deep appreciation to the following people with whom I worked on a daily basis:

**Anton Kaul** – who worked closely with me at every stage of the project and helped to write some new text material and exercises. On the many occasions that I needed mathematical or pedagogical advice, he was the person I turned to. I cannot thank him enough for his guidance and the many contributions he has made to this edition.

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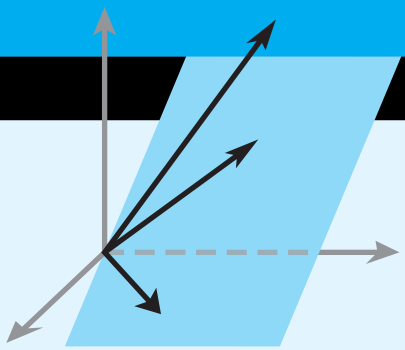
*Howard Anton*
*Chris Rorres*

# CONTENTS



CHAPTER 1

# Systems of Linear Equations and Matrices

CHAPTER CONTENTS



## INTRODUCTION

Information in science, business, and mathematics is often organized into rows and columns to form rectangular arrays called "matrices" (plural of "matrix"). Matrices often appear as tables of numerical data that arise from physical observations, but they occur in various mathematical contexts as well. For example, we will see in this chapter that all of the information required to solve a system of equations such as

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

and that the solution of the system can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs for solving systems of equations because computers are well suited for manipulating arrays of numerical information. However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a multitude of practical applications. It is the study of matrices and related topics that forms the mathematical field that we call "linear algebra." In this chapter we will begin our study of matrices.

## 1.1 Introduction to Systems of Linear Equations

Systems of linear equations and their solutions constitute one of the major topics that we will study in this course. In this first section we will introduce some basic terminology and discuss a method for solving such systems.

### Linear Equations

Recall that in two dimensions a line in a rectangular $xy$-coordinate system can be represented by an equation of the form

$$ax + by = c \quad (a, b \text{ not both } 0)$$

and in three dimensions a plane in a rectangular $xyz$-coordinate system can be represented by an equation of the form

$$ax + by + cz = d \quad (a, b, c \text{ not all } 0)$$

These are examples of "linear equations," the first being a linear equation in the variables $x$ and $y$ and the second a linear equation in the variables $x$, $y$, and $z$. More generally, we define a ***linear equation*** in the $n$ variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, a_2, \ldots, a_n$ and $b$ are constants, and the $a$'s are not all zero. In the special cases where $n = 2$ or $n = 3$, we will often use variables without subscripts and write linear equations as

$$a_1x + a_2y = b \quad (a_1, a_2 \text{ not both } 0) \tag{2}$$

$$a_1x + a_2y + a_3z = b \quad (a_1, a_2, a_3 \text{ not all } 0) \tag{3}$$

In the special case where $b = 0$, Equation (1) has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \tag{4}$$

which is called a ***homogeneous linear equation*** in the variables $x_1, x_2, \ldots, x_n$.

**EXAMPLE 1 Linear Equations**

Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions. The following are linear equations:

$$\begin{array}{ll} x + 3y = 7 & \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0 \\ \frac{1}{2}x - y + 3z = -1 & \qquad x_1 + x_2 + \cdots + x_n = 1 \end{array}$$

The following are not linear equations:

$$\begin{array}{ll} x + 3y^2 = 4 & \qquad 3x + 2y - xy = 5 \\ \sin x + y = 0 & \qquad \sqrt{x_1} + 2x_2 + x_3 = 1 \end{array}$$

A finite set of linear equations is called a ***system of linear equations*** or, more briefly, a ***linear system***. The variables are called ***unknowns***. For example, system (5) that follows has unknowns $x$ and $y$, and system (6) has unknowns $x_1$, $x_2$, and $x_3$.

$$\begin{array}{ll} 5x + y = 3 & \qquad 4x_1 - x_2 + 3x_3 = -1 \\ 2x - y = 4 & \qquad 3x_1 + x_2 + 9x_3 = -4 \end{array} \tag{5–6}$$

The double subscripting on the coefficients $a_{ij}$ of the unknowns gives their location in the system—the first subscript indicates the equation in which the coefficient occurs, and the second indicates which unknown it multiplies. Thus, $a_{12}$ is in the first equation and multiplies $x_2$.

A general linear system of $m$ equations in the $n$ unknowns $x_1, x_2, \dots, x_n$ can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{7}$$

A ***solution*** of a linear system in $n$ unknowns $x_1, x_2, \dots, x_n$ is a sequence of $n$ numbers $s_1, s_2, \dots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \dots, \quad x_n = s_n$$

makes each equation a true statement. For example, the system in (5) has the solution

$$x = 1, \quad y = -2$$

and the system in (6) has the solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$

These solutions can be written more succinctly as

$$(1, -2) \quad \text{and} \quad (1, 2, -1)$$

in which the names of the variables are omitted. This notation allows us to interpret these solutions geometrically as points in two-dimensional and three-dimensional space. More generally, a solution

$$x_1 = s_1, \quad x_2 = s_2, \dots, \quad x_n = s_n$$

of a linear system in $n$ unknowns can be written as

$$(s_1, s_2, \dots, s_n)$$

which is called an ***ordered n-tuple***. With this notation it is understood that all variables appear in the same order in each equation. If $n = 2$, then the $n$-tuple is called an ***ordered pair***, and if $n = 3$, then it is called an ***ordered triple***.

## Linear Systems in Two and Three Unknowns

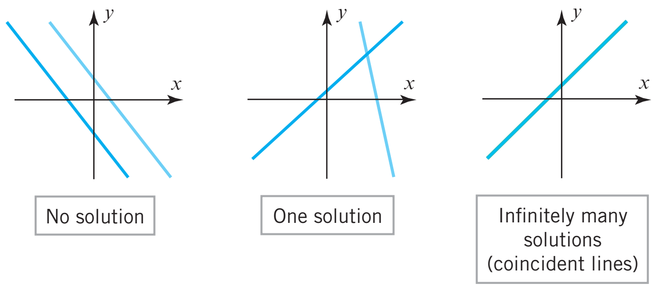
Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the $xy$-plane. Each solution $(x, y)$ of this system corresponds to a point of intersection of the lines, so there are three possibilities (Figure 1.1.1):

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

In general, we say that a linear system is ***consistent*** if it has at least one solution and ***inconsistent*** if it has no solutions. Thus, a *consistent* linear systemof two equations in

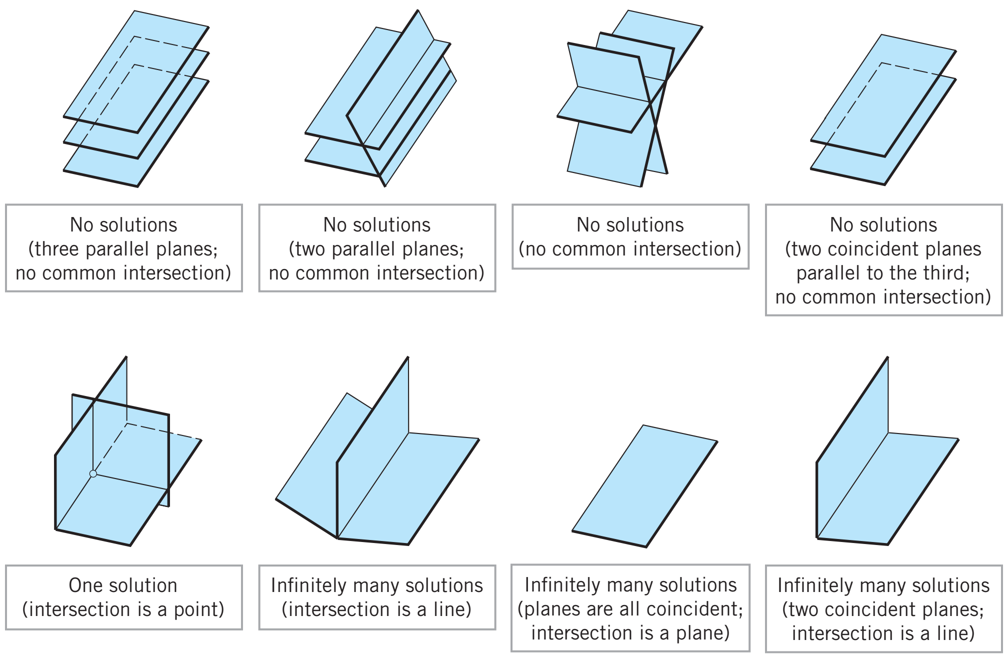



▶ Figure 1.1.1

two unknowns has either one solution or infinitely many solutions—there are no other possibilities. The same is true for a linear system of three equations in three unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities—no solutions, one solution, or infinitely many solutions (Figure 1.1.2).




▲ Figure 1.1.2

We will prove later that our observations about the number of solutions of linear systems of two equations in two unknowns and linear systems of three equations in three unknowns actually hold for *all* linear systems. That is:

*Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

### ▶ EXAMPLE 2 A Linear System with One Solution

Solve the linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

***Solution*** We can eliminate $x$ from the second equation by adding $-2$ times the first equation to the second. This yields the simplified system

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

From the second equation we obtain $y = \frac{4}{3}$, and on substituting this value in the first equation we obtain $x = 1 + y = \frac{7}{3}$. Thus, the system has the unique solution

$$x = \tfrac{7}{3}, \quad y = \tfrac{4}{3}$$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3}, \frac{4}{3}\right)$. We leave it for you to check this by graphing the lines.

### ▶ EXAMPLE 3 A Linear System with No Solutions

Solve the linear system

$$\begin{aligned} x + \phantom{3}y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

***Solution*** We can eliminate $x$ from the second equation by adding $-3$ times the first equation to the second equation. This yields the simplified system

$$\begin{aligned} x + y &= \phantom{-}4 \\ 0 &= -6 \end{aligned}$$

The second equation is contradictory, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. We leave it for you to check this by graphing the lines or by showing that they have the same slope but different $y$-intercepts.

### ▶ EXAMPLE 4 A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

***Solution*** We can eliminate $x$ from the second equation by adding $-4$ times the first equation to the second. This yields the simplified system

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restrictions on $x$ and $y$ and hence can be omitted. Thus, the solutions of the system are those values of $x$ and $y$ that satisfy the single equation

$$4x - 2y = 1 \tag{8}$$

Geometrically, this means the lines corresponding to the two equations in the original system coincide. One way to describe the solution set is to solve this equation for $x$ in terms of $y$ to obtain $x = \frac{1}{4} + \frac{1}{2}y$ and then assign an arbitrary value $t$ (called a ***parameter***)

In Example 4 we could have also obtained parametric equations for the solutions by solving (8) for $y$ in terms of $x$ and letting $x = t$ be the parameter. The resulting parametric equations would look different but would define the same solution set.

to $y$. This allows us to express the solution by the pair of equations (called ***parametric equations***)

$$x = \tfrac{1}{4} + \tfrac{1}{2}t, \quad y = t$$

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter $t$. For example, $t = 0$ yields the solution $\left(\tfrac{1}{4}, 0\right)$, $t = 1$ yields the solution $\left(\tfrac{3}{4}, 1\right)$, and $t = -1$ yields the solution $\left(-\tfrac{1}{4}, -1\right)$. You can confirm that these are solutions by substituting their coordinates into the given equations.

### ▶ EXAMPLE 5 A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} x - \phantom{2}y + 2z &= \phantom{1}5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

***Solution*** This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of $x$, $y$, and $z$ that satisfy the equation

$$x - y + 2z = 5 \tag{9}$$

automatically satisfy all three equations. Thus, it suffices to find the solutions of (9). We can do this by first solving this equation for $x$ in terms of $y$ and $z$, then assigning arbitrary values $r$ and $s$ (parameters) to these two variables, and then expressing the solution by the three parametric equations

$$x = 5 + r - 2s, \quad y = r, \quad z = s$$

Specific solutions can be obtained by choosing numerical values for the parameters $r$ and $s$. For example, taking $r = 1$ and $s = 0$ yields the solution (6, 1, 0). ◀

## Augmented Matrices and Elementary Row Operations

As the number of equations and unknowns in a linear system increases, so does the complexity of the algebra involved in finding solutions. The required computations can be made more manageable by simplifying notation and standardizing procedures. For example, by mentally keeping track of the location of the $+$'s, the $x$'s, and the $=$'s in the linear system

$$\begin{array}{cccc} a_{11}x_1 & + a_{12}x_2 & + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots + a_{2n}x_n & = b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + \cdots + a_{mn}x_n & = b_m \end{array}$$

we can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

As noted in the introduction to this chapter, the term "matrix" is used in mathematics to denote a rectangular array of numbers. In a later section we will study matrices in detail, but for now we will only be concerned with augmented matrices for linear systems.

This is called the ***augmented matrix*** for the system. For example, the augmented matrix for the system of equations

$$\begin{aligned} x_1 + \phantom{4}x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

The basic method for solving a linear system is to perform algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called ***elementary row operations*** on a matrix.

In the following example we will illustrate how to use elementary row operations and an augmented matrix to solve a linear system in three unknowns. Since a systematic procedure for solving linear systems will be developed in the next section, do not worry about how the steps in the example were chosen. Your objective here should be simply to understand the computations.

### EXAMPLE 6 Using Elementary Row Operations

In the left column we solve a system of linear equations by operating on the equations in the system, and in the right column we solve the same system by operating on the rows of the augmented matrix.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add $-2$ times the first equation to the second to obtain

Add $-2$ times the first row to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

**Maxime Bôcher (1867–1918)**

**Historical Note** The first known use of augmented matrices appeared between 200 B.C. and 100 B.C. in a Chinese manuscript entitled *Nine Chapters of Mathematical Art*. The coefficients were arranged in columns rather than in rows, as today, but remarkably the system was solved by performing a succession of operations on the columns. The actual use of the term *augmented matrix* appears to have been introduced by the American mathematician Maxime Bôcher in his book *Introduction to Higher Algebra*, published in 1907. In addition to being an outstanding research mathematician and an expert in Latin, chemistry, philosophy, zoology, geography, meteorology, art, and music, Bôcher was an outstanding expositor of mathematics whose elementary textbooks were greatly appreciated by students and are still in demand today.

[*Image: Courtesy of the American Mathematical Society www.ams.org*]

Add −3 times the first equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Add −3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add −3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add −3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by −2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by −2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add −1 times the second equation to the first to obtain

$$\begin{aligned} x \quad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add −1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution in this example can also be expressed as the ordered triple (1, 2, 3) with the understanding that the numbers in the triple are in the same order as the variables in the system, namely, $x, y, z$.

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \quad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution $x = 1$, $y = 2$, $z = 3$ is now evident. ◀

## Exercise Set 1.1

**1.** In each part, determine whether the equation is linear in $x_1$, $x_2$, and $x_3$.

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ (b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$ (d) $x_1^{-2} + x_2 + 8x_3 = 5$

(e) $x_1^{3/5} - 2x_2 + x_3 = 4$ (f) $\pi x_1 - \sqrt{2}x_2 = 7^{1/3}$

**2.** In each part, determine whether the equation is linear in $x$ and $y$.

(a) $2^{1/3}x + \sqrt{3}y = 1$ (b) $2x^{1/3} + 3\sqrt{y} = 1$

(c) $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$ (d) $\frac{\pi}{7}\cos x - 4y = 0$

(e) $xy = 1$ (f) $y + 7 = x$

**3.** Using the notation of Formula (7), write down a general linear system of

(a) two equations in two unknowns.

(b) three equations in three unknowns.

(c) two equations in four unknowns.

**4.** Write down the augmented matrix for each of the linear systems in Exercise 3.

In each part of Exercises 5–6, find a linear system in the unknowns $x_1, x_2, x_3, \ldots$, that corresponds to the given augmented matrix.

**5.** (a) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$

**6.** (a) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

In each part of Exercises 7–8, find the augmented matrix for the linear system.

**7.** (a) $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$ (b) $\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$

(c) $\begin{aligned} 2x_2 \phantom{+x_3} - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$

**8.** (a) $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 \phantom{- x_2} + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$

(c) $\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$

**9.** In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

(a) $(3, 1, 1)$ (b) $(3, -1, 1)$ (c) $(13, 5, 2)$

(d) $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$ (e) $(17, 7, 5)$

**10.** In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} x + 2y - 2z &= 3 \\ 3x - y + z &= 1 \\ -x + 5y - 5z &= 5 \end{aligned}$$

(a) $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$ (b) $\left(\frac{5}{7}, \frac{8}{7}, 0\right)$ (c) $(5, 8, 1)$

(d) $\left(\frac{5}{7}, \frac{10}{7}, \frac{2}{7}\right)$ (e) $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$

**11.** In each part, solve the linear system, if possible, and use the result to determine whether the lines represented by the equations in the system have zero, one, or infinitely many points of intersection. If there is a single point of intersection, give its coordinates, and if there are infinitely many, find parametric equations for them.

(a) $\begin{aligned} 3x - 2y &= 4 \\ 6x - 4y &= 9 \end{aligned}$ (b) $\begin{aligned} 2x - 4y &= 1 \\ 4x - 8y &= 2 \end{aligned}$ (c) $\begin{aligned} x - 2y &= 0 \\ x - 4y &= 8 \end{aligned}$

**12.** Under what conditions on $a$ and $b$ will the following linear system have no solutions, one solution, infinitely many solutions?

$$\begin{aligned} 2x - 3y &= a \\ 4x - 6y &= b \end{aligned}$$

In each part of Exercises 13–14, use parametric equations to describe the solution set of the linear equation.

**13.** (a) $7x - 5y = 3$

(b) $3x_1 - 5x_2 + 4x_3 = 7$

(c) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

(d) $3v - 8w + 2x - y + 4z = 0$

**14.** (a) $x + 10y = 2$

(b) $x_1 + 3x_2 - 12x_3 = 3$

(c) $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

(d) $v + w + x - 5y + 7z = 0$

In Exercises 15–16, each linear system has infinitely many solutions. Use parametric equations to describe its solution set.

**15.** (a) $\begin{aligned} 2x - 3y &= 1 \\ 6x - 9y &= 3 \end{aligned}$

(b) $\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 3x_1 + 9x_2 - 3x_3 &= -12 \\ -x_1 - 3x_2 + x_3 &= 4 \end{aligned}$

**16.** (a) $\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$ (b) $\begin{aligned} 2x - y + 2z &= -4 \\ 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$

In Exercises 17–18, find a single elementary row operation that will create a 1 in the upper left corner of the given augmented matrix and will not create any fractions in its first row.

**17.** (a) $\begin{bmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & -5 & 0 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

**18.** (a) $\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$